

Topological Properties of Orthostochastic Matrices*

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ABSTRACT

The set of $n \times n$ orthostochastic matrices with the topology induced by the Euclidean metric is shown to be compact and path-connected. For $n < 3$, the set of orthostochastic matrices is identical to the set of doubly stochastic matrices. In this paper, it is shown that for $n \geq 3$ the orthostochastic matrices are not everywhere dense in the set of doubly stochastic matrices, thus answering a question of L. Mirsky in his survey article on doubly stochastic matrices [2].

Denote by $\mathcal{U}(n)$ the set of $n \times n$ unitary matrices, and let $\mathcal{D}(n)$ represent the set of doubly stochastic (d.s.) matrices (those matrices with non-negative real entries and row and column sums equal to 1). Define a matrix function f from the set of $n \times n$ complex matrices to the set of $n \times n$ real matrices

$$f: C^{n \times n} \rightarrow R^{n \times n}, \quad \text{by} \quad f(A) = |a_{ij}|^2.$$

PROPOSITION. *f is continuous with respect to the usual topologies.*

Equivalently, considering $R^{n \times n}$ with the product topology, $\pi_{ij} \circ f$ is continuous for all projections π_{ij} of $R^{n \times n}$ onto R . But $(\pi_{ij} \circ f)(A) = |a_{ij}|^2 = a_{ij} \bar{a}_{ij}$, so $\pi_{ij} \circ f$ is a composition of continuous functions; hence it is continuous.

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DEFINITION. A matrix O is called orthostochastic (o.s.) if and only if there exists a unitary matrix U satisfying $(|u_{ij}|^2) = (o_{ij})$.

The set of $n \times n$ o.s. matrices $\mathcal{O}(n)$ is just the continuous image of the set of $n \times n$ unitary matrices; that is, $\mathcal{O}(n) = f(\mathcal{U}(n))$. Consequently, information about $\mathcal{O}(n)$ follows from results for $\mathcal{U}(n)$.

LEMMA. $\mathcal{U}(n)$ is a compact, path-connected subspace of $C^{n \times n}$.

Proof.

(i) *Compactness.* $\mathcal{U}(n) \subset I^{n \times n}$, the unit cube, which is compact. So it suffices to establish that $\mathcal{U}(n)$ is closed. Introduce a notation for the inner product of two columns of a complex matrix by

$$g_{ij} : C^{n \times n} \rightarrow C \quad \text{with} \quad g_{ij}(A) = \sum_{k=1}^n a_{ki} \bar{a}_{kj}.$$

Then define

$$S_{ii} = \{A \in C^{n \times n} : g_{ii}(A) = 1\} = g_{ii}^{-1}(\{1\})$$

$$S_{ij} = \{A \in C^{n \times n} : g_{ij}(A) = 0\} = g_{ij}^{-1}(\{0\}) \quad \text{for } i \neq j.$$

The singleton sets $\{0\}$ and $\{1\}$ are closed in the metric space C . Furthermore, each g_{ij} , being a composition of continuous functions, is continuous. Hence, each S_{ij} is closed, since it is just the inverse image of a closed set under a continuous mapping. From the characterization of the set of unitary matrices as those complex matrices with orthonormal columns, it follows that $\mathcal{U}(n) = \cap_{i,j} S_{ij}$. Since $\mathcal{U}(n)$ is a finite intersection of closed sets, $\mathcal{U}(n)$ is closed and therefore compact.

(ii) *Path-connectedness.* It is first shown that the set of diagonal unitary matrices is path-connected. The unit circle in the complex plane, $S^1 = \{z \in C : |z| = 1\}$, is path-connected. Therefore the product set $\prod_{i=1}^n S^1$ is also path-connected. Now the diagonal unitary matrices (the complex matrices with zero entries everywhere except for elements of unit modulus along the main diagonal) form a subspace which is homeomorphic to $\prod_{i=1}^n S^1$.

By the spectral theorem, any $U \in \mathcal{U}(n)$ is unitarily similar to a diagonal unitary matrix:

$$U = V^* D V,$$

and accordingly,

$$\begin{aligned}
 \mathfrak{U}(n) &= \{ U \in C^{n \times n} : U = V^* D V \text{ with } D, V \in \mathfrak{U}(n) \text{ and } D \text{ diagonal} \} \\
 &= \bigcup_{V \in \mathfrak{U}(n)} \left[\bigcup_{\substack{D \in \mathfrak{U}(n) \\ \text{diagonal}}} \{ V^* D V \} \right] \\
 &= \bigcup_V T_V, \quad \text{where } T_V = \bigcup_{\substack{D \in \mathfrak{U}(n) \\ \text{diagonal}}} \{ V^* D V \}.
 \end{aligned}$$

Now T_V , the continuous image of the path-connected set of diagonal unitary matrices, is path-connected. Choosing $D = I$ in the similarity transformation gives that $I \in T_V$ for all V . Consequently, as is well known, $\mathfrak{U}(n)$ is path-connected, since $\mathfrak{U}(n)$ is the union of the path-connected sets T_V , each of which contains the common point I . ■

Employing standard theorems from topology, the results for $\mathfrak{O}(n)$ are obtained readily, as illustrated below.

THEOREM 1. $\mathfrak{O}(n)$ is a compact, path-connected subspace of $R^{n \times n}$.

Proof. Since $\mathfrak{U}(n)$ is compact and path-connected, its continuous image $f(\mathfrak{U}(n)) = \mathfrak{O}(n)$ enjoys the same properties. ■

From the fact that the row and column vectors of a unitary matrix have a norm of 1, one sees that every orthostochastic matrix is doubly stochastic. In a paper on doubly stochastic matrices [2], L. Mirsky posed the question of whether the orthostochastic matrices are dense in the set of doubly stochastic matrices. This question can be answered easily once it is established that $\mathfrak{O}(n)$ is a proper subset of $\mathfrak{D}(n)$ for $n \geq 3$.

THEOREM 2.

$$\begin{aligned}
 \mathfrak{O}(n) &= \mathfrak{D}(n) && \text{for } n = 1, 2. \\
 \mathfrak{O}(n) &\neq \mathfrak{D}(n) && \text{for } n \geq 3.
 \end{aligned}$$

Proof. For $n = 1$, let $U = (1) \in \mathfrak{U}(n)$. Then $O = f(U) = (1) \in \mathfrak{O}(1)$ and

$\mathfrak{D}(1) = \{(1)\} \subset \mathfrak{O}(1)$. In the case $n=2$, let

$$U_\alpha = \begin{bmatrix} \alpha^{1/2} & (1-\alpha)^{1/2} \\ -(1-\alpha)^{1/2} & \alpha^{1/2} \end{bmatrix} \in \mathfrak{U}(2), \quad 0 \leq \alpha \leq 1,$$

then

$$O_\alpha = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix} = f(U_\alpha) \in \mathfrak{O}(2).$$

However,

$$\begin{aligned} \mathfrak{D}(2) &= \left\{ (a_{ij}) \in R^{2 \times 2} : a_{ij} \geq 0, a_{11} + a_{12} = a_{12} + a_{22} = a_{11} + a_{21} = a_{21} + a_{22} = 1 \right\} \\ &= \left\{ \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix} : 0 \leq \alpha \leq 1 \right\} \subset \mathfrak{O}(2). \end{aligned}$$

For larger n , it suffices to show that there exists a d.s. matrix which is not o.s. Consider the matrix

$$D(3) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathfrak{D}(3),$$

which is attributed to A. J. Hoffmann [1]. Let

$$U = (u_{ij}) \in f^{-1}(D(3)).$$

Then

$$g_{12}(U) = u_{11}\bar{u}_{12} + u_{21}\bar{u}_{22} + u_{31}\bar{u}_{32} = u_{31}\bar{u}_{32}$$

since $|u_{11}|^2 = |u_{22}|^2 = 0$. Since neither u_{31} nor \bar{u}_{32} is zero (as $|u_{31}|^2 = |u_{32}|^2 = \frac{1}{2}$), $g_{12}(U) \neq 0$. That is, $D(3)$ is not the image under f of any unitary matrix, hence $D(3) \notin \mathfrak{O}(3)$.

By defining

$$D(n) = \left(\begin{array}{c|c} D(3) & 0 \\ \hline 0 & I_{n-3} \end{array} \right) \in \mathfrak{D}(n),$$

analogous reasoning proves that $D(n) \notin \mathfrak{O}(n)$.

THEOREM 3. $\mathfrak{O}(n)$ is not dense in $\mathfrak{D}(n)$ for $n \geq 3$.

Proof. Suppose $\mathfrak{O}(n)$ were dense in $\mathfrak{D}(n)$, that is, $\text{Cl } \mathfrak{O}(n) = \mathfrak{D}(n)$. Theorem 1 states that $\text{Cl } \mathfrak{O}(n) = \mathfrak{O}(n)$. It follows that $\mathfrak{O}(n) = \mathfrak{D}(n)$, which contradicts Theorem 2. Therefore, $\text{Cl } \mathfrak{O}(n) \neq \mathfrak{D}(n)$ for $n \geq 3$ and the theorem is proved.

REFERENCES

- 1 M. Marcus, Some properties and applications of doubly stochastic matrices, *Am. Math. Mon.*, **67** (1960), 215–221.
- 2 L. Mirsky, Results and problems in the theory of doubly-stochastic matrices, *Z. Wahrscheinlichkeitstheorie* **1** (1963), 319–334.

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